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BIG BOUNCE AND INFLATION FROM GRAVITATIONAL FOUR-FERMION INTERACTION

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The four-fermion gravitational interaction is induced by torsion, and gets dominating on the Planck scale. The regular, axial-axial part of this interaction by itself does not stop the gravitational compression. However, the anomalous, vector-vector interaction results in a natural way both in big bounce and in inflation.

Keywords: Planck scale; gravitational four-fermion interaction; big bounce; inflation.

1. The observation that, in the presence of torsion, the interaction of fermions with gravity results in the four-fermion interaction of axial currents, goes back at least to Ref. 1.

We start our discussion of the four-fermion gravitational interaction with the analysis of its most general form.

As has been demonstrated in Ref. 2, the common action for the gravitational field can be generalized as follows:

$$S_g = -\frac{1}{16\pi G}\int d^4x\,(-e)\,e_I^\mu e_J^\nu\left(R_{\mu\nu}^{IJ} - \frac{1}{\gamma}\tilde{R}_{\mu\nu}^{IJ}\right); \tag{1}$$

here and below G is the Newton gravitational constant, $I, J = 0,1,2,3$ (and M, N below) are internal Lorentz indices, $\mu,\nu = 0,1,2,3$ are space-time indices, e_μ^I is the tetrad field, e is its determinant, and e_I^μ is the object inverse to e_μ^I. The curvature tensor is

$$R_{\mu\nu}^{IJ} = -\partial_\mu \boldsymbol{\omega}^{IJ}{}_\nu + \partial_\nu \boldsymbol{\omega}^{IJ}{}_\mu + \boldsymbol{\omega}^{IK}{}_\mu\,\boldsymbol{\omega}_K{}^J{}_\nu - \boldsymbol{\omega}^{IK}{}_\nu\,\boldsymbol{\omega}_K{}^J{}_\mu,$$

here $\boldsymbol{\omega}_\mu^{IJ}$ is the connection. The first term in (1) is in fact the common action of the gravitational field written in tetrad components.

The second term in (1), that with the dual curvature tensor

$$\tilde{R}_{\mu\nu}^{IJ} = \frac{1}{2}\,\varepsilon^{IJ}_{KL} R_{\mu\nu}^{KL}\,,$$

does not vanish in the presence of spinning particles generating torsion.

As to the so-called Barbero-Immirzi parameter γ, its numerical value

$$\gamma = 0.274 \tag{2}$$

was obtained for the first time in Ref. 3, as the solution of the "secular" equation

$$\sum_{j=1/2}^{\infty} (2j+1)e^{-2\pi\gamma\sqrt{j(j+1)}} = 1. \tag{3}$$

Interaction of fermions with gravity results, in the presence of torsion, in the four-fermion action which looks as follows:

$$S_{ff} = \frac{3}{2}\pi G \frac{\gamma^2}{\gamma^2+1} \int d^4x\sqrt{-g}\left[\eta_{IJ}A^I A^J + \frac{\alpha}{\gamma}\eta_{IJ}(V^I A^J + A^I V^J) - \alpha^2\,\eta_{IJ}V^I V^J\right]; \tag{4}$$

here and below g is the determinant of the metric tensor, A^I and V^I are the total axial and vector neutral currents, respectively:

$$A^I = \sum_a A^I_a = \sum_a \bar{\psi}_a\,\gamma^5\,\gamma^I\,\psi_a\,; \qquad V^I = \sum_a V^I_a = \sum_a \bar{\psi}_a\,\gamma^I\,\psi_a\,; \tag{5}$$

the sums over a in (5) extend over all sorts of elementary fermions with spin 1/2.

The AA contribution to expression (4) corresponds (up to a factor) to the action derived long ago in Ref. 1. Then, this contribution was obtained in the limit $\gamma \to \infty$ in Ref. 4 (when comparing the corresponding result from Ref. 4 with (4), one should note that our convention $\eta_{IJ} = \mathrm{diag}(1,-1,-1,-1)$ differs in sign from that used in Ref. 4). The present form of the AA interaction, given in (4), was derived in Ref. 5.

As to VA and VV terms in (4), they were derived in Ref. 6 as follows. The common action for fermions in gravitational field

$$S_f = \int d^4x\,\sqrt{-g}\,\frac{1}{2}\,[\,\bar{\psi}\,\gamma^I\,e^\mu_I\,i\boldsymbol{\nabla}_\mu\psi - i\,\overline{\boldsymbol{\nabla}_\mu\psi}\,\gamma^I e^\mu_I\psi] \tag{6}$$

can be generalized to:

$$S_f = \int d^4x\,\sqrt{-g}\,\frac{1}{2}\,[(1-i\alpha)\,\bar{\psi}\,\gamma^I\,e^\mu_I\,i\boldsymbol{\nabla}_\mu\psi - (1+i\alpha)\,i\,\overline{\boldsymbol{\nabla}_\mu\psi}\,\gamma^I e^\mu_I\psi]; \tag{7}$$

here $\boldsymbol{\nabla}_\mu = \partial_\mu - \frac{1}{4}\,\boldsymbol{\omega}^{IJ}{}_\mu\,\gamma_I\gamma_J\,;\;\; \boldsymbol{\omega}^{IJ}{}_\mu$ is the connection. The real constant α introduced in (7) is of no consequence, generating only a total derivative, if the theory is torsion free. However, in the presence of torsion this constant gets operative. In particular, as demonstrated in Ref. 6, it generates the VA and VV terms in the gravitational four-fermion interaction (4).

Simple dimensional arguments demonstrate that interaction (4), being proportional to the Newton constant G and to the particle number density squared, gets essential and dominates over the common interactions only at very high densities and temperatures, i.e. on the Planck scale.

The list of papers where the gravitational four-fermion interaction is discussed in connection with cosmology, is too lengthy for this short note. Therefore, I refer

here only to the most recent Ref. 7, with a quite extensive list of references. However, in all those papers the discussion is confined to the analysis of the axial-axial interaction.

In particular, in my paper Ref. 8 it was claimed that VA and VV terms in formula (4) are small as compared to the AA one. The argument was as follows. Under these extreme conditions, the number densities of both fermions and antifermions increase, due to the pair creation, but the total vector current density V^I remains intact.

By itself, this is correct. However, the analogous line of reasoning applies to the axial current density A^I. It is in fact the difference of the left-handed and right-handed axial currents: $A^I = A^I_L - A^I_R$. There is no reason to expect that this difference changes with temperature and/ or pressure.

Moreover, the fermionic number (as distinct from the electric charge) is not a long-range charge. Therefore, even the conservation of fermionic number could be in principle violated, for instance, by the decay of a neutral particle (majoron) into two neutrinos. (I am grateful to A.D. Dolgov for attracting my attention to this possibility.)

So, we work below with both currents, A and V.

2. Let us consider the energy-momentum tensor (EMT) $T_{\mu\nu}$ generated by action (4). Therein, the expression in square brackets has no explicit dependence at all either on the metric tensor, or on its derivatives. The metric tensor enters action S_{ff} via $\sqrt{-g}$ only, so that the corresponding EMT is given by relation

$$\frac{1}{2}\,\sqrt{-g}\,T_{\mu\nu} = \frac{\delta}{\delta g_{\mu\nu}} S_{ff}\,. \tag{8}$$

Thus, with identity

$$\frac{1}{\sqrt{-g}}\frac{\partial\sqrt{-g}}{\partial g^{\mu\nu}} = -\frac{1}{2}\,g_{\mu\nu}\,, \tag{9}$$

we arrive at the following expression for the EMT:

$$T_{\mu\nu} = -\frac{3\pi}{2}\frac{\gamma^2}{\gamma^2+1}\,g_{\mu\nu}\left[\eta_{IJ}A^IA^J\,+\,\frac{\alpha}{\gamma}\,\eta_{IJ}\,(V^IA^J + A^IV^J) - \alpha^2\,\eta_{IJ}V^IV^J\right],$$

or, in the tetrad components,

$$T_{MN} = -\frac{3\pi}{2}\frac{\gamma^2}{\gamma^2+1}\,\eta_{MN}\left[\eta_{IJ}A^IA^J\,+\,\frac{\alpha}{\gamma}\,\eta_{IJ}\,(V^IA^J + A^IV^J) - \alpha^2\,\eta_{IJ}V^IV^J\right]. \tag{10}$$

We note first of all that this EMT in the locally inertial frame corresponds to the equation of state

$$p = -\,\varepsilon; \tag{11}$$

here and below $\varepsilon = T_{00}$ is the energy density, and $p = T_{11} = T_{22} = T_{33}$ is the pressure.

Let us analyze the expressions for ε and p in our case of the interaction of two ultrarelativistic fermions (labeled a and b) in their locally inertial center-of-mass system.

The axial and vector currents of fermion a are, respectively,

$$A_a^I = \frac{1}{4E^2}\,\phi_a^\dagger\,\{E\,\boldsymbol{\sigma}_a\,(\mathbf{p}'+\mathbf{p}),\ (E^2-(\mathbf{p}'\mathbf{p}))\,\boldsymbol{\sigma}_a+\mathbf{p}'(\boldsymbol{\sigma}_a\mathbf{p})+\mathbf{p}\,(\boldsymbol{\sigma}_a\mathbf{p}')-i\,[\mathbf{p}'\times\mathbf{p}]\}\,\phi_a =$$

$$= \frac{1}{4}\,\phi_a^\dagger\,\{\boldsymbol{\sigma}_a\,(\mathbf{n}'+\mathbf{n}),\ (1-(\mathbf{n}'\mathbf{n}))\,\boldsymbol{\sigma}_a+\mathbf{n}'(\boldsymbol{\sigma}_a\mathbf{n})+\mathbf{n}\,(\boldsymbol{\sigma}_a\mathbf{n}')-i\,[\mathbf{n}'\times\mathbf{n}]\}\,\phi_a; \quad (12)$$

$$V_a^I = \frac{1}{4E^2}\,\phi_a^\dagger\,\{E^2+(\mathbf{p}'\mathbf{p})+i\,\boldsymbol{\sigma}_a\,[\mathbf{p}'\times\mathbf{p}],\ E\,(\mathbf{p}'+\mathbf{p}-i\,\boldsymbol{\sigma}_a\times(\mathbf{p}'-\mathbf{p}))\}\,\phi_a =$$

$$= \frac{1}{4}\,\phi_a^\dagger\,\{1+(\mathbf{n}'\mathbf{n})+i\,\boldsymbol{\sigma}_a\,[\mathbf{n}'\times\mathbf{n}],\ \mathbf{n}'+\mathbf{n}\,-i\,\boldsymbol{\sigma}_a\times(\mathbf{n}'-\mathbf{n})]\}\,\phi_a; \qquad (13)$$

here E is the energy of fermion a, $\mathbf{n}$ and $\mathbf{n}'$ are the unit vectors of its initial and final momenta $\mathbf{p}$ and $\mathbf{p}'$, respectively; under the discussed extreme conditions all fermion masses can be neglected. In the center-of-mass system, the axial and vector currents of fermion b are obtained from these expressions by changing the signs: $\mathbf{n} \to -\mathbf{n}$, $\mathbf{n}' \to -\mathbf{n}'$. Then, after averaging over the directions of $\mathbf{n}$ and $\mathbf{n}'$, we arrive at the following semiclassical expressions for the nonvanishing components of the energy-momentum tensor, i.e. for the energy density ε and pressure p (for the correspondence between ε, p and EMT components see Ref. 9, § 35):

$$\varepsilon = T_{00} = -\frac{\pi}{48}\,\frac{\gamma^2}{\gamma^2+1}\,G\sum_{a,\,b}\rho_a\,\rho_b\,[(3-11\,<\boldsymbol{\sigma}_a\boldsymbol{\sigma}_b>)-\alpha^2(60-28\,<\boldsymbol{\sigma}_a\boldsymbol{\sigma}_b>)]$$

$$= -\frac{\pi}{48}\,\frac{\gamma^2}{\gamma^2+1}\,G\,\rho^2\,[(3-11\,\zeta)-\alpha^2(60-28\,\zeta)]\,; \qquad (14)$$

$$p = T_{11} = T_{22} = T_{33} = \frac{\pi}{48}\,\frac{\gamma^2}{\gamma^2+1}\,G\sum_{a,\,b}\rho_a\,\rho_b\,[(3-11\,<\boldsymbol{\sigma}_a\boldsymbol{\sigma}_b>)$$

$$-\,\alpha^2(60-28\,<\boldsymbol{\sigma}_a\boldsymbol{\sigma}_b>)]$$

$$= \frac{\pi}{48}\,\frac{\gamma^2}{\gamma^2+1}\,G\,\rho^2\,[(3-11\,\zeta)-\alpha^2(60-28\,\zeta)]\,; \qquad (15)$$

here and below ρ_a and ρ_b are the number densities of the corresponding sorts of fermions and antifermions, $\rho = \sum_a \rho_a$ is the total density of fermions and antifermions, the summation $\sum_{a,\,b}$ extends over all sorts of fermions and antifermions; $\zeta =< \boldsymbol{\sigma}_a\boldsymbol{\sigma}_b >$ is the average value of the product of corresponding $\boldsymbol{\sigma}$-matrices, presumably universal for any a and b. Since the number of sorts of fermions and antifermions is large, one can neglect here for numerical reasons the contributions of exchange and annihilation diagrams, as well as the fact that if $\boldsymbol{\sigma}_a$ and $\boldsymbol{\sigma}_b$ refer

to the same particle, $<\boldsymbol{\sigma}_a \boldsymbol{\sigma}_b>= 3$. The parameter ζ , just by its physical meaning, in principle can vary in the interval from 0 (which corresponds to the complete thermal incoherence or to the antiferromagnetic ordering) to 1 (which corresponds to the complete ferromagnetic ordering).

It is only natural that after the performed averaging over $\mathbf{n}$ and $\mathbf{n}'$, the P-odd contributions of VA to ε and p vanish.

3. Though for $\alpha \sim 1$ the VV interaction dominates numerically the results (14) and (15), it is instructive to start the analysis with the discussion of the case $\alpha = 0$, at least, for the comparison with the previous investigations. We note in particular that, according to (14), the contribution of the gravitational spin-spin interaction to energy density is positive, i.e. the discussed interaction is repulsive for fermions with aligned spins. This our conclusion agrees with that made long ago in Ref. 4.

To simplify the discussion, we confine from now on to the region around the Planck scale, so that one can neglect effects due to the common fermionic EMT, originating from the Dirac Lagrangian and linear in the particle density ρ.

A reasonable dimensional estimate for the temperature τ of the discussed medium is

$$\tau \sim m_{\rm Pl} \tag{16}$$

(here and below $m_{\rm Pl}$ is the Planck mass). This temperature is roughly on the same order of magnitude as the energy scale ω of the discussed interaction

$$\omega \sim G\,\rho \sim m_{\rm Pl}\,. \tag{17}$$

Numerically, however, τ and ω can differ essentially. Both options, $\tau > \omega$ and $\tau < \omega$, are conceivable.

If the temperature is sufficiently high, $\tau \gg \omega$, it destroys the spin-spin correlations in formulas (14) and (15). In the opposite limit, when $\tau \ll \omega$, the energy density (14) is minimized by the antiferromagnetic spin ordering. Thus, in both these limiting cases the energy density and pressure simplify to

$$\varepsilon = -\,\frac{\pi}{16}\,\frac{\gamma^2}{\gamma^2+1}\,G\,\rho^2; \qquad p = \frac{\pi}{16}\,\frac{\gamma^2}{\gamma^2+1}\,G\,\rho^2\,. \tag{18}$$

The energy density ε, being negative and proportional to ρ^2, decreases with the growth of ρ. On the other hand, the common positive pressure p grows together with ρ. Both these effects result in the compression of the fermionic matter, and thus make the discussed system unstable.

A curious phenomenon could be possible if initially the temperature is sufficiently small, $\tau < \omega$, so that equations (18) hold. Then the matter starts compressing, its temperature increases, and the correlator $\zeta = <\boldsymbol{\sigma}_a \boldsymbol{\sigma}_b>$ could arise. When (and if!) ζ exceeds its critical value $\zeta_{cr} = 3/11$, the compression changes to expansion. Thus, we would arrive in this case at the big bounce situation.

However, I am not aware of any physical mechanism which could result here in the transition from the initial antiferromagnetic ordering to the ferromagnetic one with positive $\zeta = < \boldsymbol{\sigma}_a \boldsymbol{\sigma}_b >$.

Here one should mention also quite popular idea according to which the gravitational collapse can be stopped by a positive spin-spin contribution to the energy. However, how such spin-spin correlation could survive under the discussed extremal conditions? The naïve classical arguments do not look appropriate in this case.

4. Let us come back now to equations (14), (15). In this general case, with nonvanishing anomalous VV interaction, the big bounce takes place if the energy density (14) is positive (and correspondingly, the pressure (15) is negative). In other words, the anomalous, VV interaction results in big bounce under the condition

$$\alpha^2 \geq \frac{3 - 11\zeta}{4\,(15 - 7\zeta)}\,. \tag{19}$$

For vanishing spin-spin correlation ζ, this condition simplifies to

$$\alpha^2 \geq \frac{1}{20}\,. \tag{20}$$

The next remark refers to the spin-spin contribution to energy density (14)

$$\varepsilon_\zeta = -\frac{\pi}{48}\,\frac{\gamma^2}{\gamma^2+1}\,G\,\rho^2\,(28\,\alpha^2 - 11)\,\zeta\,. \tag{21}$$

It could result in the ferromagnetic ordering of spins if $\alpha^2 > 11/28$. Whether or not this ordering takes place, depends on the exact relation between $G\rho$ and temperature, both of which are on the order of magnitude of m_{Pl}.

5. One more comment related to equations (14), (15). As mentioned already, according to them, the equation of state, corresponding to the discussed gravitational four-fermion interaction, is

$$p = -\varepsilon. \tag{22}$$

It is rather well-known that this equation of state results in the exponential expansion of the Universe. Let us consider in this connection our problem.

We assume that the Universe is homogeneous and isotropic, and thus is described by the well-known Friedmann-Robertson-Walker (FRW) metric

$$ds^2 = dt^2 - a(t)^2[dr^2 + f(r)(d\theta^2 + \sin^2\theta\, d\phi^2)]; \tag{23}$$

here $f(r)$ depends on the topology of the Universe as a whole:

$$f(r) = r^2, \quad \sin^2 r, \quad \sinh^2 r$$

for the spatial flat, closed, and open Universe, respectively. As to the function $a(t)$, it depends on the equation of state of the matter.

The Einstein equations for the FRW metric (23) reduce to

$$H^2 \equiv \left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G\varepsilon}{3} - \frac{k}{a^2}\,, \tag{24}$$

$$\frac{\ddot{a}}{a} = -\frac{4\pi G}{3}\,(\varepsilon + 3p\,). \tag{25}$$

They are supplemented by the covariant continuity equation, which can be written as follows:

$$\dot{\varepsilon} + 3\,H(\varepsilon + p\,) = 0; \qquad H = \frac{\dot{a}}{a}\,. \tag{26}$$

For the energy-momentum tensor (14), (15), dominating on the Planck scale, and resulting in $\varepsilon = -p$, this last equation reduces to

$$\dot{\varepsilon} = 0, \qquad \text{or} \qquad \varepsilon = \text{const}. \tag{27}$$

In its turn, equation (25) simplifies to

$$\frac{\ddot{a}}{a} = \frac{8\pi G\varepsilon}{3} = \text{const}. \tag{28}$$

In this way, we arrive at the following expansion law:

$$a \sim \exp(Ht), \qquad \text{where} \qquad H = \sqrt{\frac{8\pi G\varepsilon}{3}} = \text{const} \tag{29}$$

(as usual, the second, exponentially small, solution of eq. (28) is neglected here).

Thus, the discussed gravitational four-fermion interaction results in the inflation on the Planck scale.

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